Problem 2.30

(a) Check that the results of Exs. 2.5 and 2.6, and Prob. 2.11, are consistent with Eq. 2.33.

(b) Use Gauss's law to find the field inside and outside a long hollow cylindrical tube, which carries a uniform surface charge σ. Check that your result is consistent with Eq. 2.33.

(c) Check that the result of Ex. 2.8 is consistent with boundary conditions 2.34 and 2.36.

2.4 ■ WORK AND ENERGY IN ELECTROSTATICS

2.4.1 ■ The Work It Takes to Move a Charge

Suppose you have a stationary configuration of source charges, and you want to move a test charge Q from point $\mathbf{a}$ to point $\mathbf{b}$ (Fig. 2.39). *Question:* How much work will you have to do? At any point along the path, the electric force on Q is $\mathbf{F} = Q\mathbf{E}$; the force *you* must exert, in opposition to this electrical force, is $-Q\mathbf{E}$. (If the sign bothers you, think about lifting a brick: gravity exerts a force mg *downward,* but *you* exert a force mg *upward.* Of course, you *could* apply an even greater force—then the brick would accelerate, and part of your effort would be "wasted" generating kinetic energy. What we're interested in here is the *minimum* force you must exert to do the job.) The work you do is therefore

$$W = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l} = -Q \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} = Q[V(\mathbf{b}) - V(\mathbf{a})].$$

Notice that the answer is independent of the path you take from $\mathbf{a}$ to $\mathbf{b}$; in mechanics, then, we would call the electrostatic force "conservative." Dividing through by Q, we have

$$V(\mathbf{b}) - V(\mathbf{a}) = \frac{W}{Q}. \tag{2.38}$$

In words, *the potential difference between points* $\mathbf{a}$ *and* $\mathbf{b}$ *is equal to the work per unit charge required to carry a particle from* $\mathbf{a}$ *to* $\mathbf{b}$. In particular, if you want to bring Q in from far away and stick it at point $\mathbf{r}$, the work you must do is

$$W = Q[V(\mathbf{r}) - V(\infty)],$$

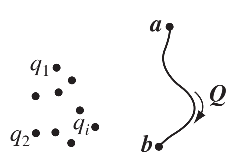

FIGURE 2.39

so, if you have set the reference point at infinity,

$$W = QV(\mathbf{r}). \tag{2.39}$$

In this sense, *potential* is potential *energy* (the work it takes to create the system) *per unit charge* (just as the *field* is the *force* per unit charge).

2.4.2 ■ The Energy of a Point Charge Distribution

How much work would it take to assemble an entire *collection* of point charges? Imagine bringing in the charges, one by one, from far away (Fig. 2.40). The first charge, q_1, takes *no* work, since there is no field yet to fight against. Now bring in q_2. According to Eq. 2.39, this will cost you $q_2 V_1(\mathbf{r}_2)$, where V_1 is the potential due to q_1, and $\mathbf{r}_2$ is the place we're putting q_2:

$$W_2 = \frac{1}{4\pi\epsilon_0} q_2 \left(\frac{q_1}{\mathscr{r}_{12}}\right)$$

($\mathscr{r}_{12}$ is the distance between q_1 and q_2 once they are in position). As you bring in each charge, nail it down in its final location, so it doesn't move when you bring in the next charge. Now bring in q_3; this requires work $q_3 V_{1,2}(\mathbf{r}_3)$, where $V_{1,2}$ is the potential due to charges q_1 and q_2, namely, $(1/4\pi\epsilon_0)(q_1/\mathscr{r}_{13} + q_2/\mathscr{r}_{23})$. Thus

$$W_3 = \frac{1}{4\pi\epsilon_0} q_3 \left(\frac{q_1}{\mathscr{r}_{13}} + \frac{q_2}{\mathscr{r}_{23}}\right).$$

Similarly, the extra work to bring in q_4 will be

$$W_4 = \frac{1}{4\pi\epsilon_0} q_4 \left(\frac{q_1}{\mathscr{r}_{14}} + \frac{q_2}{\mathscr{r}_{24}} + \frac{q_3}{\mathscr{r}_{34}}\right).$$

The *total* work necessary to assemble the first four charges, then, is

$$W = \frac{1}{4\pi\epsilon_0} \left(\frac{q_1 q_2}{\mathscr{r}_{12}} + \frac{q_1 q_3}{\mathscr{r}_{13}} + \frac{q_1 q_4}{\mathscr{r}_{14}} + \frac{q_2 q_3}{\mathscr{r}_{23}} + \frac{q_2 q_4}{\mathscr{r}_{24}} + \frac{q_3 q_4}{\mathscr{r}_{34}}\right).$$

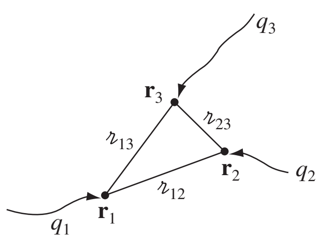

FIGURE 2.40

You see the general rule: Take the product of each pair of charges, divide by their separation distance, and add it all up:

$$W = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j>i}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}}. \tag{2.40}$$

The stipulation $j > i$ is to remind you not to count the same pair twice. A nicer way to accomplish this is *intentionally* to count each pair twice, and then divide by 2:

$$W = \frac{1}{8\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}} \tag{2.41}$$

(we must still avoid $i = j$, of course). Notice that in this form the answer plainly does not depend on the *order* in which you assemble the charges, since every pair occurs in the sum.

Finally, let's pull out the factor q_i:

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i\left(\sum_{j\neq i}^{n}\frac{1}{4\pi\epsilon_0}\frac{q_j}{\mathscr{r}_{ij}}\right).$$

The term in parentheses is the potential at point $\mathbf{r}_i$ (the position of q_i) due to all the *other* charges—all of them, now, not just the ones that were present at some stage during the assembly. Thus,

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i). \tag{2.42}$$

That's how much work it takes to assemble a configuration of point charges; it's also the amount of work you'd get back if you dismantled the system. In the meantime, it represents energy stored in the configuration ("potential" energy, if you insist, though for obvious reasons I prefer to avoid that word in this context).

Problem 2.31

(a) Three charges are situated at the corners of a square (side a), as shown in Fig. 2.41. How much work does it take to bring in another charge, $+q$, from far away and place it in the fourth corner?

(b) How much work does it take to assemble the whole configuration of four charges?

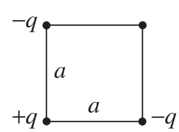

FIGURE 2.41

Problem 2.32 Two positive point charges, q_A and q_B (masses m_A and m_B) are at rest, held together by a massless string of length a. Now the string is cut, and the particles fly off in opposite directions. How fast is each one going, when they are far apart?

Problem 2.33 Consider an infinite chain of point charges, $\pm q$ (with alternating signs), strung out along the x axis, each a distance a from its nearest neighbors. Find the work per particle required to assemble this system. [*Partial Answer:* $-\alpha q^2/(4\pi\epsilon_0 a)$, for some dimensionless number α; your problem is to determine α. It is known as the **Madelung constant**. Calculating the Madelung constant for 2- and 3-dimensional arrays is much more subtle and difficult.]

2.4.3 ■ The Energy of a Continuous Charge Distribution

For a volume charge density ρ, Eq. 2.42 becomes

$$W = \frac{1}{2}\int \rho V \, d\tau. \tag{2.43}$$

(The corresponding integrals for line and surface charges would be $\int \lambda V \, dl$ and $\int \sigma V \, da$.) There is a lovely way to rewrite this result, in which ρ and V are eliminated in favor of $\mathbf{E}$. First use Gauss's law to express ρ in terms of $\mathbf{E}$:

$$\rho = \epsilon_0 \boldsymbol{\nabla}\cdot\mathbf{E}, \quad \text{so} \quad W = \frac{\epsilon_0}{2}\int (\boldsymbol{\nabla}\cdot\mathbf{E}) V \, d\tau.$$

Now use integration by parts (Eq. 1.59) to transfer the derivative from $\mathbf{E}$ to V:

$$W = \frac{\epsilon_0}{2}\left[-\int \mathbf{E}\cdot(\boldsymbol{\nabla} V)\, d\tau + \oint V\mathbf{E}\cdot d\mathbf{a}\right].$$

But $\boldsymbol{\nabla} V = -\mathbf{E}$, so

$$W = \frac{\epsilon_0}{2}\left(\int_{\mathcal{V}} E^2 \, d\tau + \oint_{\mathcal{S}} V\mathbf{E}\cdot d\mathbf{a}\right). \tag{2.44}$$

But what volume *is* this we're integrating over? Let's go back to the formula we started with, Eq. 2.43. From its derivation, it is clear that we should integrate over the region where the charge is located. But actually, any *larger* volume would do just as well: The "extra" territory we throw in will contribute nothing to the integral, since $\rho = 0$ out there. With this in mind, we return to Eq. 2.44. What happens *here,* as we enlarge the volume beyond the minimum necessary to trap all the charge? Well, the integral of E^2 can only increase (the integrand being positive); evidently the surface integral must decrease correspondingly to leave the sum intact. (In fact, at large distances from the charge, E goes like $1/r^2$ and V like $1/r$, while the surface area grows like r^2; roughly speaking, then, the surface integral goes down like $1/r$.) Please understand: Eq. 2.44 gives you the correct energy W, *whatever* volume you use (as long as it encloses all the charge), but the contribution from the volume integral goes up, and that of the surface integral goes down, as you take larger and larger volumes. In particular, why not integrate over *all* space? Then the surface integral goes to zero, and we are left with

$$\boxed{W = \frac{\epsilon_0}{2}\int E^2\,d\tau} \quad \text{(all space)}. \tag{2.45}$$

Example 2.9. Find the energy of a uniformly charged spherical shell of total charge q and radius R.

Solution 1
Use Eq. 2.43, in the version appropriate to surface charges:

$$W = \frac{1}{2}\int \sigma V\,da.$$

Now, the potential at the surface of this sphere is $(1/4\pi\epsilon_0)q/R$ (a constant—Ex. 2.7), so

$$W = \frac{1}{8\pi\epsilon_0}\frac{q}{R}\int \sigma\,da = \frac{1}{8\pi\epsilon_0}\frac{q^2}{R}.$$

Solution 2
Use Eq. 2.45. Inside the sphere, $\mathbf{E} = \mathbf{0}$; outside,

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}, \quad \text{so} \quad E^2 = \frac{q^2}{(4\pi\epsilon_0)^2 r^4}.$$

Therefore,

$$W_{\text{tot}} = \frac{\epsilon_0}{2(4\pi\epsilon_0)^2}\int_{\text{outside}} \left(\frac{q^2}{r^4}\right)(r^2 \sin\theta\,dr\,d\theta\,d\phi)$$
$$= \frac{1}{32\pi^2\epsilon_0}q^2 4\pi \int_R^\infty \frac{1}{r^2}\,dr = \frac{1}{8\pi\epsilon_0}\frac{q^2}{R}.$$

Problem 2.34 Find the energy stored in a uniformly charged solid sphere of radius R and charge q. Do it three different ways:

(a) Use Eq. 2.43. You found the potential in Prob. 2.21.

(b) Use Eq. 2.45. Don't forget to integrate over *all space*.

(c) Use Eq. 2.44. Take a spherical volume of radius a. What happens as $a \to \infty$?

Problem 2.35 Here is a fourth way of computing the energy of a uniformly charged solid sphere: Assemble it like a snowball, layer by layer, each time bringing in an infinitesimal charge dq from far away and smearing it uniformly over the surface, thereby increasing the radius. How much work dW does it take to build up the radius by an amount dr? Integrate this to find the work necessary to create the entire sphere of radius R and total charge q.

2.4.4 ■ Comments on Electrostatic Energy

(i) A perplexing "inconsistency." Equation 2.45 clearly implies that the energy of a stationary charge distribution is always *positive.* On the other hand, Eq. 2.42 (from which 2.45 was in fact derived), can be positive or negative. For instance, according to Eq. 2.42, the energy of two equal but opposite charges a distance $\mathscr{r}$ apart is $-(1/4\pi\epsilon_0)(q^2/\mathscr{r})$. What's gone wrong? Which equation is correct?

The answer is that *both* are correct, but they speak to slightly different questions. Equation 2.42 does not take into account the work necessary to *make* the point charges in the first place; we *started* with point charges and simply found the work required to bring them together. This is wise strategy, since Eq. 2.45 indicates that the energy of a point charge is in fact *infinite*:

$$W = \frac{\epsilon_0}{2(4\pi\epsilon_0)^2}\int\left(\frac{q^2}{r^4}\right)(r^2\sin\theta\,dr\,d\theta\,d\phi) = \frac{q^2}{8\pi\epsilon_0}\int_0^\infty \frac{1}{r^2}\,dr = \infty.$$

Equation 2.45 is more *complete,* in the sense that it tells you the *total* energy stored in a charge configuration, but Eq. 2.42 is more appropriate when you're dealing with point charges, because we prefer (for good reason!) to leave out that portion of the total energy that is attributable to the fabrication of the point charges themselves. In practice, after all, the point charges (electrons, say) are *given* to us ready-made; all *we* do is move them around. Since we did not put them together, and we cannot take them apart, it is immaterial how much work the process would involve. (Still, the infinite energy of a point charge is a recurring source of embarrassment for electromagnetic theory, afflicting the quantum version as well as the classical. We shall return to the problem in Chapter 11.)

Now, you may wonder where the inconsistency crept into an apparently watertight derivation. The "flaw" lies between Eqs. 2.42 and 2.43: in the former, $V(\mathbf{r}_i)$ represents the potential due to all the *other* charges *but not* q_i, whereas in the latter, $V(\mathbf{r})$ is the *full* potential. For a *continuous* distribution, there is no distinction, since the amount of charge *right at the point* $\mathbf{r}$ is vanishingly small, and its contribution to the potential is zero. But in the presence of point charges you'd better stick with Eq. 2.42.

(ii) Where is the energy stored? Equations 2.43 and 2.45 offer two different ways of calculating the same thing. The first is an integral over the charge distribution; the second is an integral over the field. These can involve completely different regions. For instance, in the case of the spherical shell (Ex. 2.9) the charge is confined to the surface, whereas the electric field is everywhere *outside* this surface. Where *is* the energy, then? Is it stored in the field, as Eq. 2.45 seems to suggest, or is it stored in the charge, as Eq. 2.43 implies? At the present stage this is simply an unanswerable question: I can tell you what the total energy is, and I can provide you with several different ways to compute it, but it is impertinent to worry about *where* the energy is located. In the context of radiation theory (Chapter 11) it is useful (and in general relativity it is *essential*) to regard the energy as stored in the field, with a density

$$\frac{\epsilon_0}{2}E^2 = \text{ energy per unit volume.} \tag{2.46}$$

But in electrostatics one could just as well say it is stored in the charge, with a density $\frac{1}{2}\rho V$. The difference is purely a matter of bookkeeping.

(iii) The superposition principle. Because electrostatic energy is *quadratic* in the fields, it does *not* obey a superposition principle. The energy of a compound system is *not* the sum of the energies of its parts considered separately—there are also "cross terms":

$$\begin{aligned} W_{\text{tot}} &= \frac{\epsilon_0}{2}\int E^2\,d\tau = \frac{\epsilon_0}{2}\int (\mathbf{E}_1 + \mathbf{E}_2)^2\,d\tau \\ &= \frac{\epsilon_0}{2}\int \left(E_1^2 + E_2^2 + 2\mathbf{E}_1\cdot\mathbf{E}_2\right)d\tau \\ &= W_1 + W_2 + \epsilon_0\int \mathbf{E}_1\cdot\mathbf{E}_2\,d\tau. \end{aligned} \tag{2.47}$$

For example, if you double the charge everywhere, you *quadruple* the total energy.

Problem 2.36 Consider two concentric spherical shells, of radii a and b. Suppose the inner one carries a charge q, and the outer one a charge $-q$ (both of them uniformly distributed over the surface). Calculate the energy of this configuration, (a) using Eq. 2.45, and (b) using Eq. 2.47 and the results of Ex. 2.9.

Problem 2.37 Find the interaction energy ($\epsilon_0\int \mathbf{E}_1\cdot\mathbf{E}_2\,d\tau$ in Eq. 2.47) for two point charges, q_1 and q_2, a distance a apart. [*Hint:* Put q_1 at the origin and q_2 on the z axis; use spherical coordinates, and do the r integral first.]

2.5 ■ CONDUCTORS

2.5.1 ■ Basic Properties

In an **insulator,** such as glass or rubber, each electron is on a short leash, attached to a particular atom. In a metallic **conductor,** by contrast, one or more electrons per atom are free to roam. (In liquid conductors such as salt water, it is ions that do the moving.) A *perfect* conductor would contain an *unlimited* supply of free charges. In real life there are no perfect conductors, but metals come pretty close, for most purposes.